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ON THE SOLUTION OF CERTAIN INTEGRAL EQUATIONS OF MIXED PROBLEMS OF PLATE BENDING THEORY*

V.B. ZELENTSOV

The problem of the bending of a Kirchhoff-Love plate in the shape of a strip under the impression of a thin linear rigid inclusion fastened at one of the edges of the plate when the other edge of the plate is rigidly clamped is considered. The problem is reduced by a Fourier integral transform to the solution of a convolution-type integral equation of the first kind in a finite segment with a regular kernel. The exact inversion of the principal part of the corresponding integral operator is constructed in the class of functions with non-integrable singularities on the segment edges. An effective asymptotic solution is given for the integral equation under investigation in this class of functions in the whole range of variation of the characteristic parameter λ. The results obtained are verified numerically. Analogous integral equations were examined in /1, 2/. The mode of investigation is similar to that proposed in /3/.

1. Formulation of the problem. A semi-infinite Kirchhoff-Love plate of width h $(|x| < \infty,\ 0 \leqslant y \leqslant h)$ is considered. The side face $(y = 0)$ is rigidly clamped while a rigid thin inclusion of length $2a$, impressed in the plate by a force P, is soldered to the other face $(y = h)$ for $|x| \leqslant a$. A function $f(x)$ describes the shape and settlement of the inclusion. The other part of the face $y = h$, outside the inclusion, is force-free. There is non normal laod on the plate. In this case the plate deflection $w(x, y)$ is described by a biharmonic equation with the boundary conditions

$$\begin{aligned}
&\Delta^2 w(x, y) = 0\\
&w(x, 0) = w_y'(x, 0) = 0,\ |x| < \infty\\
&M_y(x, h) = V_y(x, h) = 0,\ |x| > a;\\
&M_y(x, h) = 0,\ w(x, h) = f(x),\ |x| \leqslant a\\
&w(x, y) \to 0 \ \text{as}\ \sqrt{x^2 + y^2} \to \infty
\end{aligned}$$

The mixed boundary value problem (1.1) is reduced by a Fourier transformation to the solution of an integral equation of the first kind with the difference kernel

$$\lambda^2 \int_{-1}^{1} \varphi(\xi)\, d\xi \int_0^\infty K(u) \cos u \frac{x-\xi}{\lambda}\, du = \pi f(x), \quad |x| \leqslant 1 \tag{1.2}$$

$$\varphi(x) = 2a^3 (3 - 2\nu - \nu^2) D^{-1} V_y(x, h), \quad \lambda = \frac{h}{a} \tag{1.3}$$

$$K(u) = {}^1/_2 (3 - 2\nu - \nu^2)^{-1} (\operatorname{sh} 2u - 2u)(4 \operatorname{ch}^2 u + (1 - \nu)^2 u^2 - (1 + \nu)^2 \operatorname{sh}^2 u)^{-1} u^{-3}$$

Here $\varphi(x)$ is the reduced generalized transverse force in the segment $|x| \leqslant 1$, ν is Poisson's ratio, and D is the bending stiffness.

The function $K(u)$ is even and meromorphic in the complex $u = \sigma + i\tau$ plane and possesses the following asymptotic properties

$$\begin{aligned}
&K(u) = u^{-3} + O(e^{-2u}),\ u \to \infty\\
&K(u) = A_0 + O(u^2),\ u \to 0,\ A_0 = (3 - 2\nu - \nu^2)^{-1} 6^{-1}
\end{aligned} \tag{1.4}$$

2. Properties of the kernel of integral equation (1.2). We investigate the kernel of the problem under consideration

$$k(t) = \int_0^\infty K(u) \cos ut\, du, \quad t = (\xi - x) \lambda^{-1} \tag{2.1}$$

as $t \to 0$ $(\lambda \to \infty)$. Using the asymptotic properties (1.4) for the function $K(u)$, we have the following lemma.

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Lemma. The representation

$$k(t) = \frac{1}{2}t^2 \ln|t| - F(t), \qquad L(u) = u^{-3}K(u) \tag{2.2}$$

$$F(t) = \int_0^\infty \left[(L(u) - 1)\cos ut + 1 - \frac{1}{2}u^2t^2e^{-u}\right]u^{-3}\,du$$

holds for values $0 \leqslant t < \infty$ for the function $k(t)$ defined by (2.1) and (1.3).

The function $F(t)$ as an even function of the complex variable $z = t + i\tau$ is regular in the strip $|t| < \infty$, $|\tau| < 2$, where it is representable by the absolutely convergent series

$$F(t) = \sum_{k=0}^{\infty} d_k t^{2k} \tag{2.3}$$

$$d_0 = -\int_0^\infty u^{-3}L(u)\,du, \qquad d_1 = \frac{3}{4} + \frac{1}{2}\int_0^\infty u^{-1}(L(u) - 1 + e^{-u})\,du$$

$$d_k = \frac{(-1)^{k+1}}{(2k)!}\int_0^\infty u^{2k-3}(L(u) - 1)\,du, \quad k = 2, 3, \ldots$$

To prove (2.2) it is necessary to use the integral

$$\int_0^\infty \left(\cos ut - 1 + \frac{1}{2}u^2t^2e^{-u}\right)u^{-3}\,du = \frac{1}{2}t^2\ln|t| - \frac{3}{4}t^2$$

The regularity of $F(t)$ in the strip follows from the fact that $L(u) = 1 + O(e^{-2u}$ as $u \to \infty$ and the results of /4/. From the regularity of the function $F(t)(0 \leqslant t < \infty)$ its continuity that of all its derivatives follows. The representation in the form of the series (2.3) is obtained by expanding $\cos ut$ in a series in ut.

3. Structure of the general solution of the integral equation (1.2).

We note that information relative to the classes of solutions of the integral equation (1.2) is presented in /1, 2/, as well as in papers on the mixed problems of plate bending (/5–9/, etc.) Analysis of the papers mentioned suggests that the solution of the integral equation (1.2) must be sought in the class of non-integrable functions of the form

$$\varphi(x) = \omega(x)(1 - x^2)^{-3/2}, \quad \omega(x) \in H_\gamma(-1, 1), \quad \gamma > {}^1/_2$$

We first find the solution of the auxiliary equation

$$\int_{-1}^{1} \varphi(\xi)\left[\frac{1}{2}(x - \xi)^2 \ln\left|\frac{x - \xi}{\lambda}\right| - d_0\lambda^2 - d_1(x - \xi)^2\right]d\xi = \pi f(x), \qquad |x| \leqslant 1 \tag{3.1}$$

Here d_0 and d_1 are evaluated by means of (2.3). We note that the solution of (3.1) is the principal term in the asymptotic of the solution of the integral equation (1.2) for large λ. It must be taken into account that the divergent integral on the left side of (3.1) is understood in the sense of the finite part /10, 11/.

We now regularize the integral in (3.1) by introducing the function

$$\varphi^*(x) = \varphi(x) - (\alpha + \beta x)(1 - x^2)^{-3/2} \tag{3.2}$$

where α and β are constants defined from the system of equations

$$\omega(1) - \alpha - \beta = 0, \quad \omega(-1) - \alpha + \beta = 0$$

We will seek the function $\varphi^*(x)$ in the class $H_\gamma(-1, 1)$ $(0 < \gamma < 1)$.

Executing the regularization mentioned and taking the necessary quadratures in the finite part sense, we differentiate the integral equation obtained from (3.1) and (3.2) three times with respect to x while assuming that

$$f'''(x) \in H_\gamma(-1, 1) \quad (0 < \gamma < 1) \tag{3.3}$$

We consequently obtain a singular equation to determine $\varphi^*(x)$

$$\int_{-1}^{1} \frac{\varphi^*(\xi)}{\xi - x}\,d\xi = -\pi f'''(x), \qquad |x| \leqslant 1$$

whose solution is known /12/. Recalling the definition of (3.2), we find the general solution of (3.1)

$$\varphi(x) = \frac{Ax^2 + Bx + C}{(1 - x^2)^{3/2}} + \frac{1}{\pi\sqrt{1 - x^2}}\int_{-1}^{1} \frac{\sqrt{1 - t^2}\,f'''(t)}{t - x}\,dt \qquad A = P, \; B = \beta, \; C = \alpha + P \tag{3.4}$$

where A, B, C are unknown constants related in the above-mentioned manner to the α, β, P taken earlier. Tjus it is shown that under the condition (3.3) the integral equation (3.1) has a solution in the class of generalized functions K /11/, which is unique.

The constants A, B, C are related to $\varphi(x)$ by the following formulas

$$A = -\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)\,d\xi, \quad B = -\frac{1}{\pi}\int_{-1}^{1}\xi\varphi(\xi)\,d\xi - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,f'''(t)\,dt \tag{3.5}$$

$$C = -\frac{1}{\pi}\int_{-1}^{1}\xi^2\varphi(\xi)\,d\xi - \frac{3}{2}A - \frac{1}{\pi}\int_{-1}^{1}t\sqrt{1-t^2}\,f'''(t)\,dt$$

It is seen that A is the total generalized transverse force acting on a rigid inclusion, B is the total twisting moment, and C is the total magnitude of the integrable part of the transverse force $\varphi(x)$.

To determine A, B, C we multiply (3.1) by $x(1-x^2)^{-3/2}dx$ and integrate with respect to x between -1 and 1, we then proceed analogously by multiplying (3.1) by $(1-x^2)^{-3/2}dx$ and $(1-x^2)^{-1/2}dx$. Taking account of the relationships obtained in this manner and of (3.5), we determine A, B, C.

Without writing down the general formulas for finding A, B, C, we represent the solution of (3.1) in the important special case when $f(x) = 1$. It has the form

$$\varphi(x) = (Ax^2 + C)(1-x^2)^{-3/2} \tag{3.6}$$

while the constants A, B, C are determined uniquely from the system of algebraic equations

$$\begin{gathered}(^1/_2\pi - \ln 2\lambda - 2d_1)\,A + C = 0\\ (-^3/_2 + 2\ln 2\lambda + 2d_0\lambda^2 + 4d_1)\,A +\\ (-^3/_2 + \ln 2\lambda + 2d_1)\cdot C = 2,\ B = 0\end{gathered}$$

which is obtained after substituting (3.6) into (3.1) and evaluating all the quadratures.

Under additional conditions on the constants A, B, C, and thereby on the function $f(x)$ and its derivatives as well, a solution can be obtained, say, that is integrable for $x = 1$. In this case, the relationship $A + B + C = 0$, should be satisifed, and solution (3.4) takes the form

$$\varphi(x) = \frac{-Ax + C}{\sqrt{(1-x)(1+x)^3}} + \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f'''(t)}{t-x}\,dt$$

On satisfying the conditions $A + B + C = 0$ and $A - B + C = 0$, we obtain a solution integrable for $|x| = 1$

$$\varphi(x) = -\frac{A}{\sqrt{1-x^2}} + \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f'''(t)}{t-x}\,dt$$

4. Inversion formula for the integral equation (1.2). Taking the lemma into account, the integral equation (1.2) can be written in the form

$$\int_{-1}^{1}\varphi(\xi)\left[\frac{1}{2}(x-\xi)^2\ln\left|\frac{x-\xi}{\lambda}\right| - \lambda^2F\left(\frac{x-\xi}{\lambda}\right)\right]d\xi = \pi f(x), \quad |x| \leqslant 1 \tag{4.1}$$

where $F(t)$ has the form (2.3).

Theorem. When condition (3.3) is satisfied, any solution of integral equation (4.1), or equivalently, of (1.2) from the class of generalized solutions K is also a solution of the integral equation

$$\varphi(x) = \frac{Ax^2 + Bx + C}{(1-x^2)^{3/2}} + \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f'''(t)}{t-x}\,dt + \tag{4.2}$$
$$\frac{\lambda^2}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt\int_{-1}^{1}\varphi(\xi)\,F_t'''\left(\frac{t-\xi}{\lambda}\right)d\xi$$

and, conversely, while the constants A, B, C are found from the formulas

$$\begin{gathered}A = 2q\,(1 + b^{-1})(\pi\Delta)^{-1},\ B = nb^{-1}\\ C = [(b^{-1} - 2)\,m - {}^1/_2\,(11 - b)\,l]\,\Delta^{-1},\ \Delta = {}^1/_2 b + {}^1/_2 + b^{-1}\\ l = \pi^{-1}p + 2\,(\pi b)^{-1}q,\ m = -\pi^{-1}p + 2\pi^{-1}q\end{gathered} \tag{4.3}$$

$$p = \int_{-1}^{1} t\sqrt{1-t^2}\, g'''(t)\, dt, \quad q = \int_{-1}^{1} (1-t^2)^{-3/2} g(t)\, dt$$

$$n = -\pi^{-1} \int_{-1}^{1} \sqrt{1-t^2}\, g'''(t)\, dt - 2(\pi b)^{-1} \int_{-1}^{1} t (1-t^2)^{-3/2} g(t)\, dt$$

$$g(t) = f(t) + \lambda^2 \pi^{-1} \int_{-1}^{1} \varphi(\xi) F\left(\frac{t-\xi}{\lambda}\right) d\xi$$

The inversion formula (4.2) for the integral equation (4.1) is obtained from (3.4), and A, B, C are found in the same way as in Sect.3. The uniqueness of solution (4.2) results from the uniqueness of solution (3.4).

5. Solution of the integral equation (4.2) or (1.2). We seek the approximate solution of the integral equation (4.2) with functions $F(t)$ of the form (2.3) as a series in negative powers of λ /12/

$$\varphi(x) = \sum_{k=0}^{\infty} q_k(x) \lambda^{-2k} \tag{5.1}$$

Substituting (5.1) into (4.2), we obtain, by retaining three terms of the expansion in (5.1) and (2.3).

$$q_0(x) = \frac{Ax^2 + Bx + C}{(1-x^2)^{3/2}} + \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, f'''(t)}{t-x}\, dt \tag{5.2}$$

$$q_i(x) = \frac{24}{\pi^2\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\, dt \int_{-1}^{1} \varkappa_i(\xi, t)\, d\xi, \quad i = 1, 2$$

$$\varkappa_1(\xi, t) = d_2 q_0(\xi)(t-\xi)$$

$$\varkappa_2(\xi, t) = d_2 q_1(\xi)(t-\xi) + 5 d_3 q_0(\xi)(t-\xi)^3$$

The constants A, B, C are determined by the relationships (4.3). In the case $f(x) = 1$ the solution of (4.1) has the form

$$q_0(x) = \frac{Ax^2 + C}{(1-x^2)^{3/2}}, \quad q_1(x) = \frac{24 d_2 A}{\sqrt{1-x^2}} \left(x^2 - \frac{1}{2}\right) \tag{5.3}$$

$$q_2(x) = \frac{120 d_3}{\sqrt{1-x^2}} \left[Ax^4 + (4A + 3C) x^2 - \frac{19}{8} A - \frac{3}{2} C \right]$$

$$A = a_{22} (d_0 \lambda^2 \Delta)^{-1}, \quad C = -a_{21} (d_0 \lambda^2 \Delta)^{-1}, \quad \Delta = a_{11} a_{22} - a_{12} a_{21}$$

$$2 d_0 a_{11} = 2 d_0 + \left(4 d_1 - \frac{3}{2}\right) \frac{1}{\lambda^2} + 2 \ln 2\lambda \frac{1}{\lambda^2} - 3 d_2 \ln 2\lambda \frac{1}{\lambda^4} + 6 d_2 (3 - d_1) \frac{1}{\lambda^4}$$

$$2 d_0 a_{12} = \left(2 d_0 - \frac{3}{2}\right) \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \ln 2\lambda + 9 d_2 \frac{1}{\lambda^4}$$

$$a_{21} = -\ln 2\lambda - \frac{5}{2} - 2 d_1 - 24 d_2 \frac{1}{\lambda^2} + \left(36 d_2^2 - \frac{405}{2} d_3\right) \frac{1}{\lambda^4}$$

$$a_{22} = 1 - 12 d_2 \frac{1}{\lambda^2} - 135 d_3 \frac{1}{\lambda^4}$$

where $d_0 = -0.60237$, $d_1 = 0.29585$, $d_2 = 0.06647$, $d_3 = -0.00759$ in the case under consideration.

We note that the solution of integral equation (2.1) written in the form (4.1) or (4.2) is effective for large values of λ ($\lambda \gg 2$).

6. Solution of the equation for small values of the parameter λ. To obtain an effective solution of the integral equation for small λ, we follow /12/. We represent the zeroth term of the asymptotic form of the solution $\varphi(x)$ of the integral equation (1.2) for small λ in the form

$$\lambda^2 \varphi(x) = \varphi_+\left(\frac{1+x}{\lambda}\right) + \varphi_-\left(\frac{1-x}{\lambda}\right) - v\left(\frac{x}{\lambda}\right) \tag{6.1}$$

where $\varphi_\pm(x)$ is the solution of the Wiener-Hopf integral equation

$$\lambda \int_{0}^{\infty} \varphi_\pm(\xi) k(x-\xi)\, d\xi = \pi f(\lambda x \pm 1), \quad 0 \leqslant x < \infty \tag{6.2}$$

and $v(x)$ is determined from the convolution equation

$$\lambda \int_{-\infty}^{\infty} v(\xi) k(x-\xi)\, d\xi = \pi f(\lambda x), \quad |x| < \infty \tag{6.3}$$

We assume that $f(x)$ is expanded in a Fourier series and we solve (6.2) for a special right side $e^{i\eta x}$. To obtain the visible solution we approximate the function $K(u)$ by a function with similar asymptotic properties and easily factorizable. For instance, we take the expression

$$K(u)=\frac{1}{\sqrt{u^2+A^2}(u^2+B^2)}\frac{(u^2+E^2)(u^2+F^2)}{(u^2+C^2)(u^2+D^2)},\quad K(0)=\frac{E^2F^2}{AB^2C^2D^2} \tag{6.4}$$

which is most optimal from the viewpoint of approximating $K(u)$ on the real axis.

We obtain the solution of a Wiener-Hopf equation of the form (6.2) for the special right side

$$\int_0^\infty \varphi(\xi)k(x-\xi)\,d\xi=\pi e^{i\eta x},\quad 0\leqslant x<\infty \tag{6.5}$$

$$k(t)=\int_{-\infty}^{\infty} K(u)e^{iut}\,du$$

We extend this equation over the whole real axis by introducing the function $l(x)$ into the consideration /4/

$$\int_0^\infty \varphi(\xi)K(x-\xi)\,d\xi=\begin{cases}\pi e^{i\eta x}, & x\geqslant 0\\ l(x), & x<0\end{cases} \tag{6.6}$$

$$l(x)=\int_0^\infty k(x-\xi)\varphi(\xi)\,d\xi$$

We apply the Fourier integral transform to the left and right sides of (6.6), whereupon we obtain the functional equation

$$K(\alpha)\Phi_+(\alpha)=\frac{i}{\sqrt{2\pi}(\alpha+\eta)}+E_-(\alpha) \tag{6.7}$$

$$\Phi_+(\alpha)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \varphi(t)e^{i\alpha t}\,dt,\quad \pi E_-(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 l(t)e^{i\alpha t}\,dt$$

Here $\Phi_+(\alpha)$ is a regular function in the upper half-plane $\operatorname{Im}(\alpha)>\tau_1$ and $E_-(\alpha)$ is likewise in the lower half-plane $\operatorname{Im}(\alpha)<\tau_2$. The left and right sides of (6.7) are functions that are regular in the strip $|\operatorname{Im}(\alpha)|<\inf(\tau_1,\tau_2,A,B,C,D)$. We factorize the function $K(\alpha)$, i.e., we represent it in the form of the product

$$K(\alpha)=K_+(\alpha)K_-(\alpha) \tag{6.8}$$

and dividing (6.7) by $K_-(\alpha)$ we obtain

$$K_+(\alpha)\Phi_+(\alpha)=\frac{i}{\sqrt{2\pi}(\alpha+\eta)K_-(\alpha)}+\frac{E_-(\alpha)}{K_-(\alpha)}$$

The function $g(\alpha)=[\sqrt{2\pi}(\alpha+\eta)K_+(\eta)]^{-1}$ is easily factorized /12/, i.e.,

$$g(\alpha)=g_+(\alpha)+g_-(\alpha)$$

$$g_+(\alpha)=i[\sqrt{2\pi}(\alpha+\eta)K_+(\eta)]^{-1},\quad g_-(\alpha)=i[\sqrt{2\pi}(\alpha+\eta)]^{-1}[K_-^{-1}(\alpha)-K_+^{-1}(\eta)]$$

We hence have

$$K_+(\alpha)\Phi_+(\alpha)-g_+(\alpha)=g_-(\alpha)+E_-(\alpha)/K_-(\alpha) \tag{6.9}$$

The last equation defines a regular function $\Gamma(\alpha)$ in the strip

$$0<\operatorname{Im}(\alpha)<\inf(\tau_1,\tau_2,A,B,C,D,E,F)$$

Taking into account that

$$K_+(\alpha)\sim\alpha^{-1/2},\ g_+(\alpha)\sim\alpha^{-1},E_-(\alpha)K_-^{-1}(\alpha)\sim\sqrt{\alpha}e^{-\varkappa\alpha},\ \Phi_+(\alpha)\sim\sqrt{\alpha},\ g_-(\alpha)\sim\sqrt{\alpha}$$

as $\alpha\to\infty$ $(\varkappa>0)$, and remarking that the function on the left side of (6.9) decreases as α^{-1} in the regularity strip as $\operatorname{Re}(\alpha)\to\infty$, while the right side grows as $\sqrt{\alpha}$ in the same strip, by following /4/ we obtain

$$\Phi_+(\alpha)=g_+(\alpha)/K_+(\alpha)$$

Applying the inverse Fourier transform to the last relationship, we obtain

$$\varphi(x)=\frac{i}{2\pi K_+(\eta)}\int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\alpha x}}{K_+(\alpha)(\alpha+\eta)}\,d\alpha,\quad c>\operatorname{Im}(-\eta) \tag{6.10}$$

which agrees with an analogous formula in /12/, but the integral in (6.10) is understood in the generalized sense. In the case of an approximation of the form (6.4) for $K(u)$ the solution of the integral equation (6.5) has the form

$$\lambda\varphi(x) = K_+^{-1}(\eta)\left[-\frac{e^{-Ax}}{\sqrt{\pi x^3}} + \sum_{i=1}^{3}\delta_i(\alpha_i)\rho(\alpha_i, x)\right]. \tag{6.11}$$

$$\rho(t, x) = (\pi x)^{-1/2} e^{-Ax} + \sqrt{A-t}\, e^{-tx} \operatorname{erf}\sqrt{(A-t)x}$$

$$\delta_j(t) = \gamma(t)(E_j - t)^{-1}(t + t_j)^{-1}, \; \gamma(t) = (B-t)(C-t)(D-t)$$

$$E_1 = \alpha_2 = -t_3 = F, \; E_{2,3} = \alpha_1 = E, \; t_{1,2} = -\alpha_3 = i\eta$$

Other solutions corresponding to simpler approximations are easily obtained from (6.11). If $f(x) = 1$, then it is necessary to set $\eta = 0$ in (6.11).

The convolution-type integral equation on the whole real axis (6.3) is solved by using a Fourier integral transform /12/ and its solution has the form $\lambda^2 v(x) = K^{-1}(\eta) e^{-i\eta x}$, while for $\eta = 0$

$$\lambda^2 v(x) = K^{-1}(0) \tag{6.12}$$

Therefore, all the necessary formulas are obtained to comprise the principal term of the asymptotic form (6.1) of the solution of the integral equation (1.2), (4.1) for $f(x) = 1$.

Later the integral characteristic of the problem is required

$$P = \int_{-1}^{1} \varphi(\xi)\, d\xi \tag{6.13}$$

i.e., the magnitude of the force with which the inclusion is impressed into the plate. Following /12/, to calculate P by means of (6.13), we take the zeroth term of the asymptotic form of the solution for small λ in the form

$$\lambda^2\varphi(x) = \varphi_+\left(\frac{1+x}{\lambda}\right)\varphi_-\left(\frac{1-x}{\lambda}\right)v^{-1}\left(\frac{x}{\lambda}\right) \tag{6.14}$$

For a constant right side of (1.2) and (4.1), the degenerate solution $v(x)$ is given by (6.12). Then substituting (6.14) into (6.13) and understanding the expression obtained as the convolution of the Laplace transform, we have

$$P = \frac{1}{\lambda 2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\Phi(p)}{p}\right]^2 e^{2p/\lambda}\, dp, \quad \Phi(p) = K_+^{-1}(p) \tag{6.15}$$

The integral (6.13) is understood in the finite-part sense, while the Laplace integral (6.15) is understood in the generalized sense taking the behaviour of $\Phi(p) \sim p\sqrt{p}$ into account as $p \to \infty$. In the case of approximation (6.4), an explicit expression can be obtained for P in terms of λ and the constants A, B, C, D, E, F, π. It is not written down here because of its awkwardness.

λ	Formulas	$x = 0$	0.2	0.4	0.6	0.8	0.9	P
2	Collocation method	0.218	0.194	0.100	−0.176	−1.41	−5.04	—
2	(5.1),(5.3)	0.212	0.193	0.118	−0.101	−1.08	−4.02	1.89
3	(5.1),(5.3)	0.004	−0.007	−0.052	−0.182	−0.756	−2.46	0.743
5	(5.1),(5.3)	−0.034	−0.039	−0.061	−0.123	−0.394	−1.18	0.240
2	(6.1)	0.211	0.191	0.115	−0.111	−1.13	−1.90	1.90
1	(6.1)	2.53	2.49	2.34	1.82	−0.759	−8.98	10.2
0.5	(6.1)	21.2	21.2	21.0	20.2	14.6	−6.05	61.6

7. Numerical analysis of the asymptotic solutions of integral equations (1.3) obtained. It can be shown that the method of "large λ" is effective for $\lambda \geqslant 2$. and the method of "small λ" for $\lambda \leqslant 2$, while together they cover the whole range of variation of the parameter $\lambda\ (0 \leqslant \lambda < \infty)$. Juncture of the solutions should be expected for $1 \leqslant \lambda \leqslant 2$. The table shows computations performed for $\lambda = 2$ by the collocation method, the method of "large λ" (formulas (5.1) and (5.3)), the method of "small λ" (formula (6.1)). The collocation method was here used as a check on the asymptotic solutions of the "large λ" and "small λ" methods with singularities in the solution (1.2) extracted at the edges. The quadratures were evaluated by the trapezoid formula with 23 nodes in the (-1,1) range. In realizing the "small λ" method, an approximation was used for the function $K(u)$ of the kernel of an integral equation of the form (6.4) with a maximum error of 2% along the real axis. The approximation parameters in this case are $A = 0.470787$, $B = 1.79730$, $C = 1.64832$, $D = 1.41073$, $E = 0.61461$, $F = 2.89502$, where $K(0) = 0.385$. In the last column we give the integral characteristic of P. The maximum discrepancy between the results obtained by these methods of $\lambda = 2$ is 3%, and 1% in the magnitude of the force P. Moreover, the table also gives values of the reduced generalized transverse force for different λ. The phenomenon of plate separation from the inclusion that occurs in the problem becomes perceptible for $\lambda < 3$. For $\lambda > 3$ the boundary layers overlap the penetrating solution.

We note that an attempt was made /2/ to solve the integral equation

$$\int_{-1}^{1} \varphi(\xi)\left[\ln|\xi - x| + \frac{3}{2}\right] d\xi = \pi f(x), \quad |x| \leqslant 1 \tag{7.1}$$

in the class of discontinuous functions of the form (3.1). According to Sects.3 and 4, a general solution of the type (3.4) can be obtained for this equation, with the sole difference that a first-order derivative of the function $f(x)$ should be under the integral, and not the third-order derivative as in (3.4). For $f(x) = 0$ the general solution is

$$\varphi(x) = (A^* x^2 + B^* x + C)(1 - x^2)^{-3/2} \tag{7.2}$$

Substituting (7.2) into (7.1) for $f(x) = 0$ we obtain the solution of (7.1) found in /2/.

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